JHEP HYPER VERSION

Received: *July 2, 2000*, Accepted: *August 3, 2000*

SU(2) WZW D-branes and their noncommutative geometry from DBI action*

Jacek Pawełczyk

Sektion Physik, Universität München
Theresienstrasse 37 80333 München, Germany, and
Institute of Theoretical Physics, Warsaw University
Hoża 69, PL-00-681 Warsaw, Poland
E-mail: Jacek.Pawelczyk@fuw.edu.pl

Abstract: Using properties of the DBI action we find D-branes on S^3 of the radius Q_5 corresponding to the conjugacy classes of SU(2). The branes are stable due to nonzero 2-form NSNS background. In the limit of large Q_5 the dynamics of branes is governed by the non-commutative Yang-Mills theory.

Keywords: D-branes, Non-Commutative Geometry.

*Work supported in part by Polish State Committee for Scientific Research (KBN) under contract 2P 03 B03 715 (1998-2000) and the Alexander-von-Humboldt Foundation.

JHEP08(2000)006

Recently it has been discovered that in a special limit the dynamics of the matrix model is described by the non-commutative Yang-Mills theory [1]. This has sparked new interest in NCG for strings propagating in NSNS antisymmetric tensor field background [2]. Introduction of D-branes has provided deeper understanding of the role of non-commutativity [3] and it has allowed to derive conditions under which NCG starts to play dominant role in the dynamics of strings. It has also led to new understanding of the connection between quantum groups and WZW models [6, 8]. These two papers have used the standard CFT language what is a drastic bound on possible applications. In particular it does not allow to analyze RR backgrounds so much studied in the context of Maldacena's conjecture [4].

The purpose of this paper is to provide understanding of the results of [6, 8] in the more universal language then that of WZW models. The hope is that after taking this lesson one would be able to derive interesting results for more general string/M-theory backgrounds. Thus we shall describe various branes on the background of SU(2) WZW model using the D-brane effective action (DBI action) only. We shall also show how the non-commutativity appears in this approach. Methods applied here are limited to the case of large level of the SU(2) WZW model what in gravity language means large radius of the S^3.

Let us recall some of the results of [6] and [8]. D-branes in the level k SU(2) WZW model are in one-to-one correspondence with special integer conjugacy classes ghg^{-1} for some fixed h [6]. There are $k+1$ of them: two D-particles ($h = \pm 1$) and $k-1$ D2-branes corresponding to two-spheres. The n-th sphere passes through the point $\exp(i\pi n\sigma^3/k) \in \mathrm{SU}(2)$, $n = 1, \ldots, k-1$. We must also stress that D3-branes and D1-branes are excluded from this list. For large k the 2-spheres are in fact so-called fuzzy spheres [15].

The example of string theory background which involves the level k SU(2) WZW model is the near horizon limit of the F1, NS5 system (see eq. [5]). Below we write only the relevant terms

$$\begin{aligned} \frac{ds^2}{\alpha'} &= Q_5\; d\Omega_3^2 \\ \frac{H^{NSNS}}{\alpha'} &= 2Q_5\; \epsilon_3 \\ e^{2\phi} &= \text{const.}\,, \end{aligned} \tag{1}$$

where Q_5 denotes the number of NS5-branes and it is equal to the level k of the SU(2) WZW model, ϵ_3 is the volume element of the unit 3-sphere.

The effective action of the D-branes is given by DBI expression

$$S_{DBI} = -T_p \int_{\mathrm{Vol}} e^{-\phi} \sqrt{-\det[(X^*G + 2\pi\alpha' F + X^*B)_{ab}]}\,. \tag{2}$$

In the following we shall discuss classical configurations of branes embedded in S^3 of (1). Before we start to analyze equation of motion resulting from (2) we state several assumptions we make which seems to be natural here. We require the string coupling constant to be small and Q_5 to be large in which case (1) is the part of an exact string background as at this limit supergravity is the perfect description of string theory. Moreover D-branes can be described completely classically by the DBI action. We also assume that the higher order correction to the DBI action are negligible. We shall be interested here only in the S^3 part of the configuration thus it is even irrelevant if we consider IIB (as above) or IIA string. Thus some of the arguments given in our paper could be easily generalized to branes embedded in a 10d manifold of the form $S^3 \times M^7$ under the condition that the embeddings are of the product structure i.e. the induced metric, pull-back of B and F fields are of block diagonal form.

Recall that D-branes are defined to be the ends of the open strings. The string couple to the external sources (gauge A and B field) as follows

$$\exp\left[\frac{i}{2\pi\alpha'}\left(\int_{\partial\Sigma} 2\pi\alpha' X^*A + \int_{\Sigma} X^*B\right)\right]. \tag{3}$$

The example of the WZW model shows that the above formula can not be well defined globally for topologically non-trivial A and B fields. It is known that for closed strings $\partial\Sigma = 0$ the proper formula is $\exp[\frac{i}{2\pi\alpha'}\int_{\widehat{\Sigma}} \widehat{X}^*H)]$ for H being locally dB and $\widehat{X}$ is an extension of X to a 3-manifold $\widehat{\Sigma}$ such that $\partial\widehat{\Sigma} = \Sigma$. Now we consider configuration of D-brane embedded into submanifold M_D of the target space manifold M. One must repeat the above construction for the open string case [6, 7]. For the world sheet with one boundary the appropriate 3-manifold must respect $\partial\widehat{\Sigma} = \Sigma + D^2$. Rewriting the WZW model with boundary we get the proper global form of (3)[1]

$$\exp\left[\frac{i}{2\pi\alpha'}\left(-\int_{D^2} \widehat{X}^*(2\pi\alpha' F + B) + \int_{\widehat{\Sigma}} \widehat{X}^*H\right)\right], \tag{4}$$

where $\widehat{X}$ is an extension of $X(\partial\Sigma)$ to a full 2-disc D^2 such that $\widehat{X}(D^2) \subset M_D$ ($F \neq 0$ only on the D-brane manifold M_D). We stress that (4) has proper gauge invariance and for topologically trivial H it reduces to

$$\exp\left[\frac{i}{2\pi\alpha'}\left(-\int_{D^2} \widehat{X}^*(2\pi\alpha' F) + \int_{\Sigma} X^*B\right)\right] \tag{5}$$

i.e. to (3). Notice that one must be able to define B on any $\widehat{X}(D^2)$ thus we must have $[H]_{M_D} = 0$. The value of the integral (4) should not depend on the way one make the extension. This forces to put

$$\frac{i}{2\pi\alpha'}\left[\int_{C^2}(2\pi\alpha' F + B) - \frac{i}{2\pi\alpha'}\int_{C^3} H\right] = 2i\pi m\,, \tag{6}$$

[1]The minus sign in front of the first term is due to different orientation of boundary $-\partial\Sigma = \partial D^2$ in D^2 compare to Σ.

where $C^2 \in H_2(M_D)$ and $C^3 \in H_3(M, M_D)$. Here a note is necessary concerning topology of the problem. For the argument we need an exact sequence of homologies

$$\cdots \to H_3(M_D) \to H_3(M) \to H_3(M, M_D) \to H_2(M_D) \to H_2(M) \to \cdots . \qquad (7)$$

If we assume that $H_3(M_D) = H_2(M) = 0$ then all cycles of $H_3(M)$ are in $H_3(M, M_D)$. Then one can write $\int_{C^3} H = \int_{C^2} B \bmod \frac{i}{2\pi} \int_{C^3_M} H = 2\pi Q_5 m$ for $C^3_M \in H_3(M)$. Thus the quantization condition reads

$$\int_{C^2} F = 2\pi n \,, \qquad n \in \mathbb{Z} \qquad (8)$$

and n is defined modulo Q_5. This is the same as postulated in [12]. In the above we have disregarded the difference between the cycles in M and M_D and their image given by $\widehat{X}$.

D3 brane. Here we discuss the Dp-branes wrapped on the entire S^3. According to the condition $[H]_{M_D} = 0$ we see that such a wrapping is impossible. We would like to provider here a different argument based on DBI action. First one must notice that due to $[H]_{M_D} = 0$ the DBI action (2) is not well defined as B is not well defined on S^3.

In order to be more specific we concentrate on D3 brane in the background (1) and change the brane description to the dual form of the DBI action discussed e.g. in [10]. It has the same classical solutions as (2) what is the property we are interested in.

$$S_{DBI} \propto \int_{\mathrm{Vol}} e^{\phi} \sqrt{-\det[(X^*G + 2\pi\alpha'\tilde{F})_{ab}]} - \pi\alpha'\tilde{F} \wedge B \,. \qquad (9)$$

The last term come form CS part of the DBI action. Integrating it by parts we get

$$\pi\alpha'\tilde{A} \wedge H^{NSNS} \qquad (10)$$

thus the action contains only the well defined B field strength. With the H^{NSNS} background given by (1) we see that there is a U(1) charge generated on the D3 world-volume. The charge can not stay on S^3 as it is a compact space, thus it forces the brane to partially unwrap the sphere. In the case of $AdS_3 \times S^3$ the brane runs into the boundary of the AdS space. The above argument follows the baryon construction of [9].

D2 brane. Here we concentrate upon D2-brane case totally wrapped on S^3. It can be also a e.g. partially wrapped D3 brane. We analyze its equation of motion and find that contrary to the naive expectation the static brane it stable. As the indication of stability we invoke the lack of the tachyonic mode for the fluctuation of the brane.

In order to analyze the classical equations of motion we must find out the pull back of B field to the brane world-volume. On any 2-d submanifold of S^3 the B

filed is well (but not uniquely) defined. We have the freedom of changing B by an exact 2-form — in our case this will realized by choice of the solution for F_{12}. In the coordinates in which the metric on S^3 is $ds^2 = Q_5[d\phi^2 + \sin^2(\phi)d\Omega_2^2]$ we have for the chart covering $\phi = 0$

$$B = Q_5\alpha' \left(\phi - \nu - \frac{1}{2}\sin(2\phi)\right)\epsilon_2\,, \tag{11}$$

where ϵ_2 is the volume form of the unit S^2.

We shall find extrema of the DBI action corresponding to branes wrapped on S^2 given by some constant angle ϕ. The Euler-Lagrange equations are respected by

$$2\pi\alpha' F = -Q_5\alpha'(\phi - \nu)\,\epsilon_2\,, \qquad \phi(x) = \phi = \text{const.} \tag{12}$$

with all the other components of F equals zero. We also set $\nu = 0$ requiring that the charge and the tension of the $\phi = 0$ brane be zero. It is worth to note that the classical solution exists for any angle ϕ. When we apply the quantization condition (8) we get

$$Q_5\phi\ = 2\pi n\,. \tag{13}$$

We remind that n is defined only modulo Q_5.

We can compare (13) with results one gets assuming that the brane couple to some RR fields i.e. carry RR charge

$$+T_p \int e^{2\pi\alpha' F + X^* B} \wedge \bigoplus_q C_q\,, \tag{14}$$

where, $T_{\mathrm{D}p} = 1/((2\pi)^p \alpha'^{(p+1)/2} g_s)$. The background $2\pi\alpha' F + X^* B$ generates RR charge of the D$(p-2)$-brane equals to

$$T_p \int_{S^2} (2\pi\alpha' F + X^* B) = -T_p(4\pi\alpha' Q_5)\frac{1}{2}\sin(2\phi)\,.$$

One expects that this charge is integer multiple of $T_{(p-2)}$ i.e.

$$\frac{1}{2\pi\alpha'}\int_{S^2}(2\pi\alpha' F + X^* B) = -2\pi\, n \tag{15}$$

but this is in contradiction with (13) for finite Q_5. If one takes the $Q_5 \to \infty$ limit then both formulae agree.[2]

The second derivative of the DBI action with respect to the gauge fields and $\phi(x)$ gives kinematics of fluctuations. One easily finds that fluctuations of $\phi(x)$ only are massive but $\phi(x)$ mixes with gauge field F leading to some massless modes [12]. Thus there is no tachyon in the spectrum and the brane configuration is stable.

[2]The gap between (13) and (15) has been filled recently in [13].

Non-commutative geometry. We can also claim that at the $Q_5 \to \infty$ limit some of the branes are described by the non-commutative geometry. Here we follow the route of [3]. First we notice that at the $Q_5 \to \infty$ limit we have

$$
\begin{aligned}
ds^2 &= \alpha' \frac{(n\pi)^2}{Q_5} d\Omega_2^2 \to 0 \\
2\pi\alpha' F + X^* B &= -\alpha'(n\pi)\,\epsilon_2 \,.
\end{aligned} \tag{16}
$$

Thus the closed string metric goes to zero while induced $2\pi\alpha' F + X^* B$ is constant on the D2-branes world-volume what is a good sign for the non-commutativity. Next we calculate the open string metric and the Poisson structure inverting $2\pi\alpha' F + X^*(B + g)$. The inverse matrix is

$$
\left(\frac{1}{X^*(g+B) + 2\pi\alpha' F}\right) = \frac{1}{Q_5 \sin\phi\alpha'} \begin{pmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{pmatrix} . \tag{17}
$$

Inverse of its symmetric part is the open string metric. We have $G_{ab} = \alpha' Q_5 \delta_{ab}$. Hence from the open string point of view all spheres have the same area! This, of course, is directly related to the flat direction $\phi =$ const. in the solution (12). The Poisson structure on S^2 (also called deformation parameter) is

$$
\Theta^{12} = \frac{2\pi}{Q_5} \cot\phi \to \frac{2}{n} \,. \tag{18}
$$

The symplectic structure is the inverse of the Poisson structure and it is $\omega_{12} = (n/2)$. One can check that this parameter precisely corresponds to the symplectic structure used by Berezin in order to quantize S^2 [14]. The non-commutative version on this S^2 is called the fuzzy spheres [15]. From [16] one may claim that the Y-M theory on this sphere is a theory of $(n+1) \times (n+1)$ hermitian matrices. Such a Y-M theory has $(n+1)^2$ degrees of freedom. Here we must stress that these results are in full agreement with [8]. It would be interesting to make explicit comparison of the brane dynamics and the above matrix model.

We conclude that the branes dynamics is described by the non-commutative Y-M theory. The branes world-volume are 2-spheres which are non-commutative manifolds with the non-commutativity parameter $\Theta^{12} = 2/n$.

A note added. Some of the results of this paper have been independently obtained in the recent paper [12].

Acknowledgments

I am grateful S. Theisen for illuminating discussions and reading the manuscript. I also thank K. Gawędzki, A. Alekseev, G. Arutyunov and A. Recknagel for comments and interest in this work.

References

[1] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165];
A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031];
F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, *J. High Energy Phys.* **02** (1999) 016 [hep-th/9810072];
H. García-Compeán, *On the deformation quantization description of matrix compactifications*, *Nucl. Phys.* **B 541** (1999) 651 [hep-th/9804188];
C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219];
V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[4] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[5] J. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *J. High Energy Phys.* **12** (1998) 005 [hep-th/9804085].

[6] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[7] K. Gawędzki, *Conformal field theory: a case study*, hep-th/9904145.

[8] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040].

[9] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[10] C. Schmidhuber, *D-brane actions*, *Nucl. Phys.* **B 467** (1996) 146 [hep-th/9601003];
A.A. Tseytlin, *Self-duality of Born-Infeld action and Dirichlet 3-brane of type-IIB superstring theory*, *Nucl. Phys.* **B 469** (1996) 51 [hep-th/9602064].

[11] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, hep-th/9907189.

[12] C. Bachas, M. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[13] W. Taylor, *D2-branes in B-fields*, *J. High Energy Phys.* **07** (2000) 039 [hep-th/0004141].

[14] F.A.Berezin, *General concept of quantization*, *Comm. Math. Phys.* **40** (1975) 153.

[15] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69.

[16] H. Grosse, C. Klimcik and P. Presnajder, *Towards finite quantum field theory in non-commutative geometry*, *Int. J. Theor. Phys.* **35** (1996) 231–244 [hep-th/9505175].